SEMICLASSICAL ANALYSIS OF GENERAL SECOND ORDER ELLIPTIC OPERATORS ON BOUNDED DOMAINS

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ABSTRACT. In this work we ascertain the semiclassical behavior of the fundamental energy and the ground state of an arbitrary second order elliptic operator, not necessarily selfadjoint, on a bounded domain. Our analysis provides us with substantial improvements of many previous results found in the context of quantum mechanics for C^∞ perturbations of the Laplacian.

1. INTRODUCTION

In this paper we analyze the limiting behavior as $h \downarrow 0$ of the principal eigenpair to the following linear boundary value problem:

$$(1.1) \qquad h^2 L\varphi + V(x)\varphi = \sigma\varphi \quad \text{in } D, \qquad \varphi = 0 \quad \text{on } \partial D,$$

where D is a bounded domain of $\mathbf{R}^N$ with boundary ∂D of class $C^{0,1}$ regularity, L is a second order uniformly elliptic operator of the form

$$(1.2) \qquad L = -\sum_{i,j=1}^{N} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{N} b_i(x)\partial_i + c(x),$$

with

$$(1.3) \qquad a_{ij} \in C(\bar{D}), \qquad b_i, c \in L^\infty(D), \qquad i, j \in \{1, ..., N\},$$

and $V \in L^\infty(D)$.

When $D = \mathbf{R}^N$, this problem has been analyzed in the context of quantum mechanics in the special cases where $L = -\Delta$ and $V \geq 0$ is a C^∞ potential bounded away from zero at infinity and having a finite number of non-degenerate zeros (see [8]-[11], [22]-[25], and the references therein), and in the context of reaction-diffusion equations in some special cases where L is selfadjoint or reducible to a selfadjoint operator, [18], [5]. Some partial results were also given in Proposition 17.3 of [12].

In quantum physics $h \simeq 6.624 \times 10^{-27}\, erg/sec$ is Planck's constant. Except at atomic scales h is very small, the length of the scales over which quantum effects are important being dependent on it. This fact provides us with a way of transition from classical to quantum mechanics, by comparing the classical limit when $h = 0$ where all quantum effects have been neglected with the original quantum system through the semiclassical regime, where the Planck constant is assumed to be arbitrarily

small. When $V \geq 0$ is a C^∞ potential bounded away from zero at infinity and having a finite number of non-degenerate zeros, then

$$(1.4) \qquad \sigma_1[-h^2\Delta + V] = E_1\,h + O(h^2) \qquad \text{as} \quad h \downarrow 0\,,$$

where $\sigma_1[-h^2\Delta + V]$ is the fundamental energy of $-h^2\Delta + V$ and E_1 is the fundamental energy of the associated harmonic oscillator obtained by localization at the wells of V. Moreover, the ground states concentrate into the set of zeros of V, and either there is a rapid eigenvalue degeneracy or the limiting ground states reside asymptotically in a single well (cf. Theorem 1.2 of [23]). It was the intention of Simon (see [22], p. 296) to study the case where V vanishes on a manifold, but to the best of our knowledge, he only observed that if some of the zeros of V are degenerate then $\sigma_1[-h^2\Delta + V]$ may go to zero faster than linearly, and that under enough hypotheses on the form of V at the degenerate minima, it should be possible to get lower bounds for that decay. Some partial answers to these problems were given by Martínez and Rouleux in [19] and by one of the authors in [16]. In the context of Problem (1.1), it was shown in [16] that if the coefficients of the operator are Hölder continuous, $V \geq 0$, and $D_0 := V^{-1}(0)$ is a smooth domain, then

$$(1.5) \qquad \lim_{h\downarrow 0} \frac{\sigma_1^D[h^2 L + V]}{h^2} = \sigma_1^{D_0}[L]\,.$$

Hereafter, given a subdomain $D_1 \subset D$ and an uniformly elliptic operator $\mathcal{L}$ in D_1, $\sigma_1^{D_1}[\mathcal{L}]$ stands for the principal eigenvalue of $\mathcal{L}$ in D_1 subject to homogeneous Dirichlet boundary conditions. In the special case when $L = -\Delta$, it was shown by one of the authors in [5] that in order to have (1.5) it suffices that D_0 supports a function of $W_0^{1,2}(D)$. In fact, if any such function with support in D_0 is trivial, then

$$(1.6) \qquad \lim_{h\downarrow 0} \frac{\sigma_1^D[h^2 L + V]}{h^2} = \infty\,.$$

In case (1.5) the fundamental energy decays quadratically, and in case (1.6) it decays at least linearly but slower than quadratically. In [19] it was shown that if $V \in C^\infty(\mathbf{R}^N)$ satisfies $V \geq 0$, $\liminf_{\|x\|\to\infty} V(x) > 0$, $V^{-1}(0) = 0$ and for $x \simeq 0$

$$C^{-1}\|x\|^\alpha \leq V(x) \leq C\|x\|^\alpha\,,$$

where $C > 0$ and $\alpha \geq 4$, then the decay rate of $\sigma_1[-h^2\Delta + V]$ as $h \downarrow 0$ is $h^{\frac{2\alpha}{\alpha+2}}$. In fact, an asymptotic expansion was shown to exist for this special case.

The main goal of this paper is ascertaining the exact decay rate of the lowest eigenpair of (1.1) for some rather general classes of potentials in L^∞. Our analysis will help to realize the nature and relationships between the several results obtained separately in the literature. We should point out that in our general setting we will have to deal with two serious difficulties. The first one comes from the lack of the symmetry of the differential operator, as we are working with general second order elliptic operators which in general do not possess a variational structure. The second one is inherent in the lack of regularity of the potential, under which we cannot hope to have the nice formal asymptotic expansions available in the literature for the case of smooth potentials by means of W.K.B. solutions. An extensive study of these asymptotic expansions can be found in the pioneering works by Helffer and Sjöstrand [9], [10], [11], which started a whole industry in this area. As general

references for these and other related problems we refer to the books of Helffer [8] and Hislop and Sigal [14].

Our motivation to consider the general setting (1.1) comes originally from the analysis of singular perturbation results in the context of reaction-diffusion equations, [5], [16], [18], and from the analysis of the stability of the trivial state for a linear wave equation with damping, [17]. When diffusivities and transport effects are uniformly small there naturally arises the problem of analyzing the decay rate of the principal eigenvalue and the behavior of the principal eigenfunctions, since they will eventually provide us with the behavior of the positive solutions for some general classes of non-linear singular perturbation problems.

The content of this paper is as follows. In Section 2 we recall some properties of principal eigenvalues and eigenfunctions that we are going to use throughout this work. We point out that some of them are new within our general setting. In fact, these properties are based upon a characterization of the strong maximum principle by means of the existence of positive strict supersolutions, which is a generalization of a result of [16] (cf. [1]), to which we refer for further details and references. In Section 3 we show that

$$(1.7) \qquad \lim_{h \downarrow 0} \sigma_1^D[h^2 L + V] = \operatorname*{ess\,inf}_D V ,$$

and that the limit of the ground states is concentrated in the region

$$\{\, x \in D \,:\, V(x) > \operatorname*{ess\,inf}_D V \,\} ,$$

i.e., we obtain the lowest order terms of the asymptotic expansion of the principal eigenpair of (1.1). These results are substantial improvements of Proposition 17.3 in the book of Hess [12] and of some other results found by the authors themselves (cf. [5] and [18]) for selfadjoint operators. Even in the simplest situations when $V^{-1}(\operatorname{ess\,inf}_D V)$ possesses several connected components, the ground states may reside asymptotically in only one of those components, instead of being asymptotically distributed in all of them (cf. Helffer [8], Helffer and Sjöstrand [9] and Simon [25]). Among other things, in Section 4 we show that the decay rates found by Martínez and Rouleux in [19] are still valid for our general setting. To be precise, let $a_{ij} \in W^{1,\infty}(D)$, $b_i, c \in L^\infty(D)$, $V \in L^\infty(\mathbf{R}^N)$,

$$\{\, x \in D \,:\, V(x) = V_L \,\} = \{x_0, ..., x_p\} , \qquad p \geq 0 ,$$

and for each $i = 0, ..., p$ let

$$V(x) = V_L + \|x - x_i\|^{\alpha_i} g_i(\omega) + o(\|x - x_i\|^{\alpha_i}) \qquad \text{as} \ \ x \to x_i , \qquad \omega = \frac{x - x_i}{\|x - x_i\|} ,$$

with $\alpha_i > 0$ such that

$$\alpha := \alpha_0 > \max_{1 \leq i \leq p} \alpha_i ,$$

where each $g_i : S^{N-1} \to \mathbf{R}_+$ is bounded and positive (bounded away from zero), S^{N-1} stands for the $(N-1)$-dimensional unit sphere, and

$$\liminf_{\|x\| \to \infty} V(x) > V_L.$$

Then we prove that

$$\lim_{h \downarrow 0} \frac{\sigma_1^D[h^2 L + V] - V_L}{h^{\frac{2\alpha}{\alpha+2}}} = s(-\sum_{i,j=1}^N a_{ij}(x_0)\partial_i\partial_j + \|y\|^\alpha g_0(\omega)) > 0 ,$$

where, for a Schrödinger operator $\mathcal{L}$ in the whole of $\mathbf{R}^N$, $s(\mathcal{L})$ stands for the spectral bound of $\mathcal{L}$. Moreover, if Ω_h stands for the principal eigenfunction associated with $\sigma_1^D[h^2 L + V]$ normalized so that $\|\Omega_h\|_{L^2(D)} = 1$, then

$$\lim_{h \downarrow 0} \|h^{\frac{N}{\alpha+2}} \Omega_h(x_0 + h^{\frac{2}{\alpha+2}} \cdot) - \Psi_0(\cdot)\|_{L^2(\mathbf{R}^N)} = 0,$$

where Ψ_0 is the principal eigenfunction associated with

$$s(-\sum_{i,j=1}^N a_{ij}(x_0)\partial_i\partial_j + \|y\|^\alpha g_0(\omega))$$

normalized so that $\int_{\mathbf{R}^N} \Psi_0^2(y)\,dy = 1$. We should point out that the corresponding results of Martínez and Rouleux were shown to hold for the very special case when $L = -\Delta$, $V \in C^\infty$, $p = 0$ and $\alpha \geq 4$. Notice that the decay rate $\frac{2\alpha}{\alpha+2}$ approaches 2 as $\alpha \to \infty$. Hence, it is rather natural to predict that whenever the potential vanishes on an open subset of D the decay rate to V_L of the fundamental energy will be h^2. This analysis will be accomplished in Section 5, where it will be shown that

$$\lim_{h \downarrow 0} \frac{\sigma_1^D[h^2 L + V] - V_L}{h^2} = \sigma_1^{D_0}[L],$$

where

$$D_0 = \{\, x \in D \;:\; V(x) = V_L \,\},$$

and that the limiting ground state is concentrated within D_0. To the best of our knowledge, this last result is completely unknown in the quantum mechanics literature.

2. The maximum principle. Main properties of the principal eigenvalues

In this section we give an extension of Theorem 2.5 in [16] to cover our setting here, and then we infer some basic properties of principal eigenvalues which will be used throughout this paper. We will use the natural product order on $L^p(D) \times L^p(\partial D)$. Recall that $p > N$ implies $W^{2,p}(D) \subset C^{2-\frac{N}{p}}(\bar{D})$ with compact embedding, and that each $u \in W^{2,p}(D)$ is a.e. twice classically differentiable in D (e.g., Theorem VIII.1 of [26]).

Suppose that $p > N$. Then $u \in W^{2,p}(D)$ is said to be strongly positive if $u(x) > 0$ for $x \in D$ and $\partial_n u(x) < 0$ for all $x \in \partial D$ with $u(x) = 0$, where n is the outward unit normal on ∂D. The operator L is said to satisfy the strong maximum principle in D if $p > N$, $u \in W^{2,p}(D)$, and $(Lu, u) > (0, 0)$ in $L^p(D) \times L^p(\partial D)$ imply that u is strongly positive. Consider the eigenvalue problem

$$(2.1) \qquad Lu = \sigma u \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D,$$

in $W^{2,p}(D)$ and let L_p denote the closure of the operator $L|_{W_0^{2,p}(D)}$ in $L^p(D)$. Then (2.1) can be reformulated as the eigenvalue equation

$$(2.2) \qquad L_p u = \sigma u \qquad \text{in } L^p(D).$$

It is an easy consequence of standard elliptic regularity theory that the spectrum and the eigenspaces of L_p are independent of $p > N$. Moreover, from the strong maximum principle and the generalization of the Krein-Rutman Theorem of [21] together with Theorem 3 in [20], the following result holds (cf. Section 2 of [1]).

Theorem 2.1. *There exists a least eigenvalue of* (2.2), *denoted by* $\sigma_1^D[L]$ *and called the principal eigenvalue of* L *in* D. *This eigenvalue is simple and possesses a unique eigenfunction, up to multiplicative constants, which can be taken positive, the so-called principal eigenfunction of* L *in* D. *Moreover, the principal eigenfunction is strongly positive, and* $\sigma_1^D[L]$ *is the only eigenvalue of* (2.2) *possessing a positive eigenfunction. Furthermore, any other eigenvalue* σ *of* (2.2) *satisfies*

$$\mathrm{Re}\ \sigma > \sigma_1^D[L],$$

and $(L_p + \lambda)^{-1} \in \mathcal{L}(L^p(D))$ *is positive, compact and irreducible for* $\lambda > \sigma_1^D[L]$.

If $p > N$, a function $\bar{u} \in W^{2,p}(D)$ is said to be a positive strict supersolution of L in D if $\bar{u} \geq 0$ and $(L\bar{u}, \bar{u}) > (0,0)$ in $L^p(D) \times L^p(\partial D)$. From the strong maximum principle it is easily seen that any positive strict supersolution is strongly positive. Moreover, the following characterization of the strong maximum principle holds (cf. Theorem 2.5 in [16] and Theorem 2.4 in [1]).

Theorem 2.2. *The following assertions are equivalent:*

(i) $\sigma_1^D[L] > 0$;

(ii) L *possesses a positive strict supersolution in* D;

(iii) L *satisfies the strong maximum principle in* D.

From this characterization we can readily get the following properties of $\sigma_1^D[L]$ which will be used throughout this work. For selfadjoint operators, these properties are easily obtained from the variational characterization of the principal eigenvalue.

Theorem 2.3. *(i) Monotonicity with respect to the potential: Let* V_1, $V_2 \in L^\infty(D)$ *be such that* $V_1 \leq V_2$ *and* $V_1 < V_2$ *on a set of positive measure. Then*

$$(2.3) \qquad \sigma_1^D[L + V_1] < \sigma_1^D[L + V_2].$$

(ii) If D_1 *is a proper subdomain of* D *with* ∂D_1 *of class* $C^{0,1}$, *then*

$$(2.4) \qquad \sigma_1^{D_1}[L] > \sigma_1^D[L].$$

Proof. (i) Let φ_1 be the principal eigenfunction associated with $\sigma_1^D[L + V_1]$. Then

$$(L + V_2)\varphi_1 = \sigma_1^D[L + V_1]\varphi_1 + (V_2 - V_1)\varphi_1 > \sigma_1^D[L + V_1]\varphi_1$$

on a set of positive measure, and hence φ_1 is a positive strict supersolution of $L + V_1 - \sigma_1^D[L + V_1]$. Thus, thanks to Theorem 2.2, we find that

$$\sigma_1^D[L + V_1 - \sigma_1^D[L + V_1]] > 0.$$

This relation implies (2.3).

(ii) Let φ denote the principal eigenfunction associated with $\sigma_1^D[L]$. Then

$$(L - \sigma_1^D[L])\varphi = 0$$

in D_1 and $\varphi > 0$ on ∂D_1. Thus, φ is a positive strict supersolution of $L - \sigma_1^D[L]$ in D_1, and hence it follows from Theorem 2.2 that

$$\sigma_1^{D_1}[L - \sigma_1^D[L]] > 0.$$

This relation implies (2.4). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. LIMITING BEHAVIOR OF THE PRINCIPAL EIGENPAIR

The following result shows the behavior of the principal eigenvalue.

Theorem 3.1. *Assume (1.3) and $V \in L^\infty(D)$. Then,*

$$(3.1) \qquad \lim_{h \downarrow 0} \sigma_1^D[h^2 L + V] = V_L \,,$$

where

$$V_L := \operatorname*{ess\,inf}_D V \,.$$

Proof. By the monotonicity of the principal eigenvalue with respect to the potential we have

$$\sigma_1^D[h^2 L + V] \geq h^2 \sigma_1^D[L] + V_L$$

and hence

$$(3.2) \qquad \liminf_{h \downarrow 0} \sigma_1^D[h^2 L + V] \geq V_L \,.$$

Since $V \in L^\infty(D)$, $V \in L^1(D)$. Hence, by the Lebesgue differentiation theorem, for almost every point $x_0 \in D$ the following holds:

$$\lim_{h \downarrow 0} \frac{1}{\mu(B_h(x_0))} \int_{B_h(x_0)} |V(x) - V(x_0)| \, dx = 0 \,,$$

where $B_h(x_0)$ denote the ball of radius h centered at x_0 and $\mu(\cdot)$ is the Lebesgue measure in $\mathbf{R}^N$. Thus, by a change of variable, for each $R > 0$ we have

$$\lim_{h \downarrow 0} \int_{B_R} |V(x_0 + hx) - V(x_0)| \, dx = 0 \qquad \text{for almost all } x_0 \in \mathbf{R}^N \,,$$

where $B_R := B_R(0)$. Therefore, for each $p \in [1, \infty)$

$$(3.3) \qquad \lim_{h \downarrow 0} \int_{B_R} |V(x_0 + hx) - V(x_0)|^p \, dx = 0 \qquad \text{for almost all } x_0 \in \mathbf{R}^N \,,$$

since $V \in L^\infty(D)$. Now, fix $\delta > 0$ arbitrary and $p > N$. By (3.3) there exists $x_0 \in D$ such that

$$(3.4) \qquad \lim_{h \downarrow 0} \int_{B_R} |V(x_0 + hx) - V(x_0)|^p \, dx = 0 \quad \text{and} \quad V(x_0) \leq V_L + \delta \,.$$

By rescaling the spatial variable, we find that

$$\sigma_1^{B_R}\left[-\sum_{i,j=1}^N a_{ij}(x_0)\partial_i\partial_j + V(x_0)\right] = \sigma_1^{B_1}\left[-\frac{1}{R^2}\sum_{i,j=1}^N a_{ij}(x_0)\partial_i\partial_j + V(x_0)\right].$$

Moreover,

$$\sigma_1^{B_1}\left[-\frac{1}{R^2}\sum_{i,j=1}^N a_{ij}(x_0)\partial_i\partial_j + V(x_0)\right]$$

$$= \frac{1}{R^2}\sigma_1^{B_1}\left[-\sum_{i,j=1}^N a_{ij}(x_0)\partial_i\partial_j\right] + V(x_0) \to V(x_0)$$

as $R \uparrow \infty$, and hence

$$\lim_{R\uparrow\infty} \sigma_1^{B_R}[-\sum_{i,j=1}^{N} a_{ij}(x_0)\partial_i\partial_j + V(x_0)] = V(x_0).$$

Fix $R = R(\delta)$ such that

$$(3.5) \qquad \sigma_1^{B_R}[-\sum_{i,j=1}^{N} a_{ij}(x_0)\partial_i\partial_j + V(x_0)] \leq V(x_0) + \frac{\delta}{2}.$$

Now, for $h > 0$ sufficiently small let $\mathcal{L}_h$ denote the differential operator

$$\mathcal{L}_h := -\sum_{i,j=1}^{N} a_{ij}(x_0 + hx)\partial_i\partial_j + h\sum_{i=1}^{N} b_i(x_0 + hx)\partial_i + h^2 c(x_0 + hx) + V(x_0 + hx)$$

with domain

$$\mathcal{D}(\mathcal{L}_h) := W_0^{1,p}(B_R) \cap W^{2,p}(B_R).$$

We claim that these operators converge as $h \downarrow 0$ to the operator

$$\mathcal{L}_0 := -\sum_{i,j=1}^{N} a_{ij}(x_0)\partial_i\partial_j + V(x_0), \qquad \mathcal{D}(\mathcal{L}_0) := W_0^{1,p}(B_R) \cap W^{2,p}(B_R),$$

in the generalized sense of Kato ([15], IV.2.4). Indeed, for each $u \in W^{2,p}(B_R)$ we have

$$(3.6) \qquad \begin{aligned} \|\mathcal{L}_h u - \mathcal{L}_0 u\|_{L^p(B_R)} \leq (&\max_{1\leq i,j\leq N} \|a_{ij}(x_0 + h\cdot) - a_{ij}(x_0)\|_{L^\infty(B_R)} \\ &+ h\max_{1\leq i\leq N} \|b_i\|_{L^\infty(D)} + h^2\|c\|_{L^\infty(D)})\|u\|_{W^{2,p}(B_R)} \\ &+ \|u\|_{L^\infty(B_R)}\|V(x_0 + h\cdot) - V(x_0)\|_{L^p(B_R)}, \end{aligned}$$

where we have used the embedding $W^{2,p}(B_R) \subset C(\bar{B}_R)$, which is compact. Moreover, since $a_{ij} \in C(\bar{D})$, for h sufficiently small $a_{ij}(x_0 + hx)$ is uniformly close to $a_{ij}(x_0)$ in B_R. Therefore, it follows from (3.4) and (3.6) that for any $\epsilon > 0$ there exists $h_0 > 0$ such that

$$\|\mathcal{L}_h u - \mathcal{L}_0 u\|_{L^p(B_R)} \leq \epsilon\|u\|_{W^{2,p}(B_R)}$$

for all $h \in (0, h_0)$ and $u \in W^{2,p}(B_R)$. This shows the convergence of the graphs of the $\mathcal{L}_h$ to the graph of $\mathcal{L}_0$, and hence the previous claim is shown. Therefore, due to Kato [15], IV.3.5, any finite system of eigenvalues of $\mathcal{L}_h$ changes with h continuously towards the corresponding eigenvalues of $\mathcal{L}_0$ just as in the finite-dimensional case. In particular, due to (3.5), for h small we have

$$\sigma_1^{B_R}[\mathcal{L}_h] \leq V(x_0) + \delta.$$

Rescaling the spatial variables, this relation gives

$$\sigma_1^{B_{R_h}(x_0)}[h^2 L + V(x)] \leq V(x_0) + \delta \leq V_L + 2\delta,$$

and therefore, the monotonicity of the principal eigenvalue with respect to the domain gives

$$\sigma_1^{D}[h^2 L + V(x)] \leq \sigma_1^{B_{R_h}(x_0)}[h^2 L + V(x)] \leq V_L + 2\delta.$$

As this can be done for any $\delta > 0$, we find that

$$\limsup_{h \downarrow 0} \sigma_1^D[h^2 L + V(x)] \leq V_L \,.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.1 provides us with the asymptotic behavior of the mapping $\lambda \to \sigma_1^D[L + \lambda V]$ at $\pm\infty$. Indeed,

$$\lim_{\lambda \uparrow \infty} \frac{\sigma_1^D[L + \lambda V]}{\lambda} = \operatorname{ess} \inf_D V \,, \qquad \lim_{\lambda \downarrow -\infty} \frac{\sigma_1^D[L + \lambda V]}{\lambda} = -\operatorname{ess} \sup_D V \,.$$

Thus, if in addition we assume

$$\operatorname{ess} \inf_D V < 0 \,, \qquad \operatorname{ess} \sup_D V > 0 \,, \qquad \sigma_1^D[L] > 0 \,,$$

then due to the concavity of $\lambda \to \sigma_1^D[L + \lambda V]$ we find that the weighted boundary value problem

$$L\varphi = \lambda V \varphi \quad \text{in } D \,, \qquad \varphi = 0 \quad \text{on } \partial D \,,$$

possesses two principal eigenvalues; one negative and the other positive. This result slightly improves the main theorem of Hess and Kato [13], obtained for $V \in C(\bar{D})$.

The following result shows that the principal eigenfunction associated with $\sigma_1^D[h^2 L + V]$ converge in $L^2(D)$ to 0 in the region $\{x \in D \ : \ V(x) > V_L\}$ as $h \downarrow 0$.

Theorem 3.2. *Suppose $a_{ij} \in W^{1,\infty}(D) = C^{0,1}(\bar{D})$ and $b_i, c, V \in L^\infty(D)$. Let $\Omega_h > 0$ denote the principal eigenfunction associated with $\sigma_1^D[h^2 L + V]$, normalized so that*

$$(3.7) \qquad\qquad\qquad\qquad \int_D |\nabla \Omega_h|^2 = 1 \,.$$

Then, for any sequence $h_n \downarrow 0$, as $n \uparrow \infty$, there exists a subsequence (relabeled by n) such that

$$\lim_{n \to \infty} \Omega_{h_n} = \Omega_0 \qquad \text{in } L^2(D) \,,$$

for some $\Omega_0 \in L^2(D)$ such that

$$\Omega_0 = 0 \qquad \text{in} \qquad D_+ := \{x \in D \ : \ V(x) > V_L \} \,.$$

In particular,

$$(3.8) \qquad\qquad\qquad\qquad \lim_{h \downarrow 0} \|\Omega_h\|_{L^2(D_+)} = 0 \,.$$

Proof. By Theorem 3.1 we already know that

$$(3.9) \qquad\qquad\qquad\qquad \lim_{h \downarrow 0} \sigma_1^D[h^2 L + V] = V_L \,.$$

If we set

$$(3.10) \qquad\qquad\qquad B_j := \sum_{i=1}^N \partial_i(a_{ij}) + b_j \,, \qquad j = 1, ..., N \,,$$

the equation

$$h^2 L\Omega_h + V\Omega_h = \sigma_1^D[h^2 L + V] \, \Omega_h$$

becomes

(3.11)

$$-h^2 \sum_{i,j=1}^{N} \partial_i(a_{ij}\partial_j\Omega_h) + h^2 \sum_{j=1}^{N} B_j\partial_j\Omega_h + h^2 c\Omega_h + V\Omega_h = \sigma_1^D[h^2 L + V]\,\Omega_h\,.$$

Multiplying (3.11) by Ω_h and integrating on D yields

(3.12)

$$h^2 \sum_{i,j=1}^{N} \int_D a_{ij}\,\partial_i\Omega_h\,\partial_j\Omega_h + h^2 \sum_{j=1}^{N} \int_D B_j\,\Omega_h\partial_j\Omega_h + \int_D (h^2 c + V)\Omega_h^2$$

$$= \sigma_1^D[h^2 L + V]\int_D \Omega_h^2\,.$$

By (3.6), (Ω_h) is bounded in $W_0^{1,2}(D)$. Thus, from any sequence (h_n), $n \geq 1$, $h_n \downarrow 0$, we can extract a subsequence, relabeled by n, such that

$$\lim_{n\to\infty} \|\Omega_{h_n} - \Omega_0\|_{L^2(D)} = 0\,,$$

for some $\Omega_0 \in L^2(D)$. Since $B_j \in L^\infty(D)$ and L is uniformly elliptic in D, particularizing (3.12) at h_n, passing to the limit as $n \to \infty$ and using (3.9) gives

$$\int_D V\Omega_0^2 = V_L \int_D \Omega_0^2\,.$$

Therefore, $\Omega_0 = 0$ in D_+. This completes the proof. □

Remark 3.3. (a) If $V \in C(D)$ and $|D \setminus D_+| = 0$, then (3.8) implies

$$\lim_{h\downarrow 0} \|\Omega_h\|_{L^2(D)} = 0,$$

and hence, due to (3.7), no subsequence of Ω_h can converge in $W_0^{1,2}(D)$ to the L^2-limit of the ground states.

(b) When V is continuous, a variant of Theorem 3.1 is to normalize Ω_h so that

$$\|\Omega_h\|_{L^\infty(D)} = 1\,.$$

Then, $\Omega_h \downarrow 0$ uniformly on any compact subset of $\bar{D} \cap D_+$. Indeed, if not, we rescale and use a blow-up argument near a point x_h where $\Omega_h(x_h) \geq a > 0$. If $x_h \to y_0$ as $h \to 0$, we obtain a solution of

$$-\sum_{i,j=1}^{N} a_{ij}(y_0)\partial_i\partial_j w + V(y_0)w = V_L\,w\,,$$

on either $\mathbf{R}^N$ or a half-space, which is bounded and non-negative and satisfies $w(y_0) \geq a > 0$. Moreover, we can use a change of variables so that $a_{ij}(y_0) = \delta_{ij}$ (cf. [7]). In both cases $w = 0$ is the unique bounded non-negative solution of this equation. This contradiction completes the proof. The result is well known for the full space case; it can be proved for instance by obtaining a linear differential equation for the average of w on spheres. In the half space case one can show that $\partial_j w > 0$ provided the half-space is $x_j > 0$ (by a variant of a result in [6]), and then show that $\lim_{x_j\to\infty} w$ satisfies a full space equation in one lower dimension.

4. DECAY RATE FOR POTENTIALS WITH SEVERAL DEGENERATE WELLS

The main result of this section is the following.

Theorem 4.1. *Suppose $a_{ij} \in W^{1,\infty}(D)$, b_i, $c \in L^\infty(D)$, $V \in L^\infty(\mathbf{R}^N)$,*

(4.1) $$\{\, x \in D \,:\, V(x) = V_L \,\} = \{x_0, ..., x_p\}, \qquad p \geq 0,$$

and for each $i = 0, ..., p$,

(4.2)
$$V(x) = V_L + \|x - x_i\|^{\alpha_i} g_i(\omega) + o(\|x - x_i\|^{\alpha_i}) \qquad as \;\; x \to x_i, \qquad \omega = \frac{x - x_i}{\|x - x_i\|},$$

for some $\alpha_i > 0$ such that

$$\alpha := \alpha_0 > \max_{1 \leq i \leq p} \alpha_i,$$

where each $g_i : S^{N-1} \to \mathbf{R}_+$ is bounded and positive (bounded away from zero). Here S^{N-1} stands for the $(N-1)$-dimensional unit sphere. Also, suppose

(4.3) $$\liminf_{\|x\| \to \infty} V(x) > V_L.$$

Then

(4.4) $$\lim_{h \downarrow 0} \frac{\sigma_1^D[h^2 L + V] - V_L}{h^{\frac{2\alpha}{\alpha+2}}} = \sigma_1^{\mathbf{R}^N}\Big[-\sum_{i,j=1}^N a_{ij}(x_0)\partial_i\partial_j + W\Big] > 0,$$

where W is the potential defined by

(4.5) $$W(y) = \|y\|^\alpha g_0(\omega), \qquad \omega = \frac{y}{\|y\|}, \qquad y \in \mathbf{R}^N.$$

Moreover, if Ω_h is the principal eigenfunction associated with $\sigma_1^D[h^2 L + V]$ normalized so that

(4.6) $$\|\Omega_h\|_{L^2(D)} = 1,$$

then

(4.7) $$\lim_{h \downarrow 0} \|h^{\frac{N}{\alpha+2}}\Omega_h(x_0 + h^{\frac{2}{\alpha+2}} \cdot) - \Psi_0(\cdot)\|_{L^2(\mathbf{R}^N)} = 0,$$

where Ψ_0 stands for the principal eigenfunction associated with

$$\sigma_1^{\mathbf{R}^N}\Big[-\sum_{i,j=1}^N a_{ij}(x_0)\partial_i\partial_j + W\Big]$$

normalized so that

$$\int_{\mathbf{R}^N} \Psi_0^2(y)\, dy = 1.$$

Proof. Thanks to Theorem 3.1, without loss of generality we can assume that

$$V_L = 0.$$

By the definition of Ω_h we have

(4.8)
$$-h^2 \sum_{i,j=1}^N a_{ij}\partial_i\partial_j\Omega_h + h^2 \sum_{j=1}^N b_j\partial_j\Omega_h + h^2 c\Omega_h + V\Omega_h = \sigma_1^D[h^2 L + V]\,\Omega_h$$

in D, and $\Omega_h = 0$ on ∂D. Thus, making the change of variable

(4.9) $\qquad y = h^{\frac{-2}{\alpha+2}}(x - x_0), \qquad \Psi_h(y) = h^{\frac{N}{\alpha+2}}\Omega_h(x), \qquad x \in D,$

we see that the new function $\Psi_h(y)$ satisfies

(4.10)

$$-\sum_{i,j=1}^{N} a_{ij}(x_0 + h^{\frac{2}{\alpha+2}}y)\partial_i\partial_j\Psi_h + h^{\frac{2}{\alpha+2}}\sum_{j=1}^{N} b_j(x_0 + h^{\frac{2}{\alpha+2}}y)\partial_j\Psi_h$$

$$+ h^{\frac{4}{\alpha+2}}c(x_0 + h^{\frac{2}{\alpha+2}}y)\Psi_h + h^{\frac{-2\alpha}{\alpha+2}}V(x_0 + h^{\frac{2}{\alpha+2}}y)\Psi_h = \frac{\sigma_1^D[h^2 L + V]}{h^{\frac{2\alpha}{\alpha+2}}}\Psi_h,$$

in

$$D_h := \{\, h^{\frac{-2}{\alpha+2}}(x - x_0) \; : \; x \in D \,\},$$

and $\Psi_h = 0$ on ∂D_h. Moreover, from (4.6) and (4.9) the following holds:

(4.11) $\qquad \displaystyle\int_{D_h} \Psi_h^2(y)\, dy = 1,$

Let $\mathcal{L}_h$ denote the differential operator defined by the left hand side of (4.10), i.e.

$$\mathcal{L}_h = -\sum_{i,j=1}^{N} a_{ij}(x_0 + h^{\frac{2}{\alpha+2}}y)\partial_i\partial_j + h^{\frac{2}{\alpha+2}}\sum_{j=1}^{N} b_j(x_0 + h^{\frac{2}{\alpha+2}}y)\partial_j$$

(4.12) $\qquad + h^{\frac{4}{\alpha+2}}c(x_0 + h^{\frac{2}{\alpha+2}}y) + h^{\frac{-2\alpha}{\alpha+2}}V(x_0 + h^{\frac{2}{\alpha+2}}y),$

with domain

$$\mathcal{D}(\mathcal{L}_h) = W_0^{1,p}(D_h) \cap W^{2,p}(D_h).$$

By the uniqueness of the principal eigenvalue, (4.10) implies

(4.13) $\qquad \sigma_1^{D_h}[\mathcal{L}_h] = \dfrac{\sigma_1^D[h^2 L + V]}{h^{\frac{2\alpha}{\alpha+2}}}.$

Fix $R > 0$ arbitrarily. Then, for h sufficiently small, B_R is a proper subdomain of D_h and hence it follows from (4.13) that

(4.14) $\qquad \dfrac{\sigma_1^D[h^2 L + V]}{h^{\frac{2\alpha}{\alpha+2}}} < \sigma_1^{B_R}[\mathcal{L}_h].$

Moreover, it follows from (4.2) that

(4.15) $\qquad h^{\frac{-2\alpha}{\alpha+2}}V(x_0 + h^{\frac{2}{\alpha+2}}y) = \|y\|^\alpha g_0(\omega) + o(1) \qquad$ as $h \downarrow 0$

uniformly in $y \in B_R$. Thus, by the same argument of the proof of Theorem 2.1 the operators $\mathcal{L}_h$ converge in B_R as $h \downarrow 0$ to the operator $\mathcal{L}_0$ defined by

(4.16)

$$\mathcal{L}_0 := -\sum_{i,j=1}^{N} a_{ij}(x_0)\partial_i\partial_j + W(y), \qquad \mathcal{D}(\mathcal{L}_0) := W_0^{1,p}(B_R) \cap W^{2,p}(B_R),$$

in the sense of [15], IV.2.4, for any $p \geq 2$. Hence,

$$\lim_{h \downarrow 0} \sigma_1^{B_R}[\mathcal{L}_h] = \sigma_1^{B_R}[\mathcal{L}_0],$$

and therefore, we find from (4.14) that

$$(4.17) \qquad \limsup_{h \downarrow 0} \frac{\sigma_1^D[h^2 L + V]}{h^{\frac{2\alpha}{\alpha+2}}} \leq \sigma_1^{B_R}[\mathcal{L}_0].$$

Set

$$(4.18) \qquad s(\mathcal{L}_0) := \inf \{ \operatorname{Re} \lambda \ : \ \lambda \in \sigma(\mathcal{L}_0) \}.$$

By formula (1.2) of [2], $s(\mathcal{L}_0)$ admits the following variational characterization:

$$s(\mathcal{L}_0) = \inf \{ \int_{\mathbf{R}^N} (\langle A\nabla\varphi, \nabla\varphi \rangle + W\varphi^2), \ \varphi \in \mathcal{C}_0^\infty(\mathbf{R}^N), \ \|\varphi\|_{L^2} = 1 \},$$

where $A = (a_{ij}(x_0))$ and $\langle \cdot, \cdot \rangle$ stands for the Euclidean product in $\mathbf{R}^N$. Therefore, since (4.17) is valid for any $R > 0$, it follows from the variational characterization of $\sigma_1^{B_R}[\mathcal{L}_0]$ that

$$(4.19) \qquad \limsup_{h \downarrow 0} \frac{\sigma_1^D[h^2 L + V]}{h^{\frac{2\alpha}{\alpha+2}}} \leq s(\mathcal{L}_0).$$

In particular, the quotients

$$\frac{\sigma_1^D[h^2 L + V]}{h^{\frac{2\alpha}{\alpha+2}}}$$

are uniformly bounded above. Now, we show that these quotients are also uniformly bounded below. Multiplying (4.10) by Ψ_h, integrating in D_h, applying the formula of integration by parts and using (4.11)–(4.13) gives

(4.20)

$$\sum_{i,j=1}^N \int_{D_h} a_{ij}(x_0 + h^{\frac{2}{\alpha+2}} y) \partial_i \Psi_h \partial_j \Psi_h + h^{\frac{2}{\alpha+2}} \sum_{j=1}^N \int_{D_h} B_j(x_0 + h^{\frac{2}{\alpha+2}} y) \Psi_h \partial_j \Psi_h$$

$$+ h^{\frac{4}{\alpha+2}} \int_{D_h} c(x_0 + h^{\frac{2}{\alpha+2}} y) \Psi_h^2 + h^{\frac{-2\alpha}{\alpha+2}} \int_{D_h} V(x_0 + h^{\frac{2}{\alpha+2}} y) \Psi_h^2 = \sigma_1^{D_h}[\mathcal{L}_h],$$

where the new coefficients B_j are those given by (3.10). By (4.11) and the Hölder inequality we find that

$$(4.21) \qquad |\sum_{j=1}^N \int_{D_h} B_j(x_0 + h^{\frac{2}{\alpha+2}} y) \Psi_h \partial_j \Psi_h| \leq \beta \|\nabla \Psi_h\|_{L^2(D_h)},$$

where

$$\beta := \sup_{x \in D} \|B(x)\|.$$

Thus, by (4.13), (4.19) and (4.21) it follows from (4.20) that for all $\epsilon > 0$ there exists $h_0 = h(\epsilon) > 0$ such that for $h \in (0, h_0)$ we have

$$\nu \|\nabla \Psi_h\|_{L^2(D_h)}^2 - \beta h^{\frac{2}{\alpha+2}} \|\nabla \Psi_h\|_{L^2(D_h)}$$

$$+ h^{\frac{-2\alpha}{\alpha+2}} \int_{D_h} V(x_0 + h^{\frac{2}{\alpha+2}} y) \Psi_h^2 \leq s(\mathcal{L}_0) + \epsilon - c_L h^{\frac{4}{\alpha+2}},$$

where $c_L := \operatorname{ess\,inf} c$ and $\nu > 0$ is the ellipticity constant of L. Since the right hand side of this inequality is bounded above uniformly in $h \in (0, h_0)$, it is easily seen

that there exists a constant $K > 0$ such that

(4.22)
$$\|\nabla \Psi_h\|_{L^2(D_h)} \leq K\,, \qquad h^{\frac{-2\alpha}{\alpha+2}} \int_{D_h} V(x_0 + h^{\frac{2}{\alpha+2}} y) \Psi_h^2 \leq K\,, \qquad \forall h \in (0, h_0)\,.$$

Moreover, going back to (4.20) and using (4.22), the same argument as above also shows that $\sigma_1^{D_h}[\mathcal{L}_h]$ are uniformly bounded below. We now show that

(4.23)
$$\lim_{h\downarrow 0} \sigma_1^{D_h}[\mathcal{L}_h] = \lim_{h\downarrow 0} \frac{\sigma_1^{D}[h^2 L + V]}{h^{\frac{2\alpha}{\alpha+2}}} = s(\mathcal{L}_0)\,.$$

To see (4.23) we argue by contradiction. By (4.19), if (4.23) is not true, then there exist $\lambda_0 \in \mathbf{R}$ and a sequence (h_n), $n \geq 1$, such that $\lim_{n\to\infty} h_n = 0$ and

(4.24)
$$\lim_{n\to\infty} \sigma_1^{D_{h_n}}[\mathcal{L}_{h_n}] = \lambda_0 < s(\mathcal{L}_0)\,.$$

By (4.22), we can assume that (h_n) has been chosen so that

(4.25)
$$\lim_{n\to\infty} \Psi_{h_n} = \Psi_0 \in W^{1,2}(\mathbf{R}^N) \;\; \text{weakly in } W^{1,2}(\mathbf{R}^N) \text{ and strongly in } L^2(\mathbf{R}^N)\,.$$

We claim that Ψ_0 is a weak solution of

(4.26)
$$-\sum_{i,j=1}^N a_{ij}(x_0)\partial_i\partial_j\Psi_0 + W\Psi_0 = \lambda_0\Psi_0\,.$$

To show this, pick $\varphi \in C_0^\infty(\mathbf{R}^N)$. Multiplying the equation of Ψ_{h_n} by φ, integrating on D_{h_n} and applying the formula of integration by parts gives

(4.27)
$$\sum_{i,j=1}^N \int_{D_{h_n}} a_{ij}(x_0 + h_n^{\frac{2}{\alpha+2}} y)\partial_i\Psi_{h_n}\partial_j\varphi$$
$$+ h_n^{\frac{2}{\alpha+2}} \sum_{j=1}^N \int_{D_{h_n}} B_j(x_0 + h_n^{\frac{2}{\alpha+2}} y)\varphi\partial_j\Psi_{h_n}$$
$$+ h_n^{\frac{4}{\alpha+2}} \int_{D_{h_n}} c(x_0 + h_n^{\frac{2}{\alpha+2}} y)\varphi\Psi_{h_n} + h_n^{\frac{-2\alpha}{\alpha+2}} \int_{D_{h_n}} V(x_0 + h_n^{\frac{2}{\alpha+2}} y)\varphi\Psi_{h_n}$$
$$= \sigma_1^{D_{h_n}}[\mathcal{L}_{h_n}] \int_{D_{h_n}} \varphi\Psi_{h_n}\,.$$

Thus, passing to the limit as $n \to \infty$ in (4.27), it follows from (4.2), (4.24) and (4.25) that

$$\sum_{i,j=1}^N \int_{\mathbf{R}^N} a_{ij}(x_0)\partial_i\Psi_0\partial_j\varphi + \int_{\mathbf{R}^N} \|y\|^\alpha g_0(\omega)\varphi\Psi_0 = \lambda_0 \int_{\mathbf{R}^N} \varphi\Psi_0\,.$$

Therefore, (4.26) holds true. Moreover, since $\lambda_0 < s(\mathcal{L}_0)$, (4.18) implies that

(4.28)
$$\Psi_0 = 0\,.$$

On the other hand, given $\delta > 0$ arbitrary, there exists $\epsilon > 0$ such that

(4.29)
$$h^{\frac{-2\alpha}{\alpha+2}} V(x_0 + h^{\frac{2}{\alpha+2}} y) \geq \delta \qquad \forall y \in \hat{D}_h := D_h \setminus \bigcup_{i=0}^p B_\epsilon(y_i)\,,$$

where

(4.30) $$y_i = h^{\frac{-2}{\alpha+2}}(x_i - x_0), \qquad i = 0, ..., p.$$

Moreover, (4.22) implies

(4.31)

$$h^{\frac{-2\alpha}{\alpha+2}} \int_{\hat{D}_h} V(x_0 + h^{\frac{2}{\alpha+2}}y)\Psi_h^2 + \sum_{i=0}^{p} h^{\frac{-2\alpha}{\alpha+2}} \int_{B_\epsilon(y_i)} V(x_0 + h^{\frac{2}{\alpha+2}}y)\Psi_h^2 \leq K,$$

uniformly in $h \in (0, h_0)$. Hence,

$$\int_{\hat{D}_h} \Psi_h^2 \leq \frac{K}{\delta},$$

and if we choose δ sufficiently large, it follows from (4.11) that

$$\sum_{i=0}^{p} \int_{B_\epsilon(y_i)} \Psi_{h_n}^2 \geq \frac{1}{2},$$

for n sufficiently large. Thus, for some $i \in \{0, ..., p\}$ we have

(4.32) $$\int_{B_\epsilon(y_i)} \Psi_{h_n}^2 \geq \frac{1}{2p}.$$

We claim that $i = 0$. Indeed, since the sequence $\Psi_{h_n}(y - y_i)$ is bounded in $W^{1,2}(B_\epsilon(0))$, without loss of generality we can assume that

$$\lim_{n\to\infty} \Psi_{h_n}(y - y_i) = \hat{\Psi}_0$$

strongly in $L^2(B_\epsilon(0))$. Moreover, by (4.32), $\hat{\Psi}_0 \neq 0$. If $i \neq 0$, then for $y \neq y_i$ we have

$$\lim_{n\to\infty} h_n^{\frac{-2\alpha}{\alpha+2}} V(x_0 + h_n^{\frac{2}{\alpha+2}}(y - y_i)) = \infty,$$

since $\|x - x_i\| \leq \epsilon h^{\frac{2}{\alpha+2}}$ and $\alpha > \alpha_i$. Moreover, if $0 < \nu < \epsilon$, then

$$h_n^{\frac{-2\alpha}{\alpha+2}} \int_{B_\epsilon(0)\setminus B_\nu(0)} V(x_0 + h_n^{\frac{2}{\alpha+2}}(y - y_i))\Psi_{h_n}^2(y - y_i) \leq K,$$

and so passing to the limit as $n \to \infty$ gives

$$\int_{B_\epsilon(0)\setminus B_\nu(0)} \hat{\Psi}_0^2 = 0,$$

and therefore $\hat{\Psi}_0 = 0$ a.e. on $B_\epsilon(0)$, which is impossible. This contradiction shows that $i = 0$. Now, (4.32) gives

$$\int_{B_\epsilon(0)} \Psi_{h_n}^2 \geq \frac{1}{2p},$$

and, passing to the limit as $n \to \infty$, we find from (4.25) that

$$\int_{B_\epsilon(0)} \Psi_0^2 \geq \frac{1}{2p}.$$

This relation contradicts (4.28) and therefore shows (4.23). Observe that the previous argument also shows (4.7), since $s(\mathcal{L}_0)$ is a simple eigenvalue of $\mathcal{L}_0$ in $L^2(\mathbf{R}^N)$

and the argument works out along any sequence $h_n \downarrow 0$. We should point out that in the statement of the theorem we have used the notation

$$\sigma_1^{\mathbf{R}^N}[\mathcal{L}_0] = s(\mathcal{L}_0).$$

It remains to show that $s(\mathcal{L}_0) > 0$. This follows easily from (4.20). Indeed, passing to the limit as $n \to \infty$ and using Fatou's lemma, we find that

$$\int_{\mathbf{R}^N} (\langle A\nabla\Psi_0, \Psi_0 \rangle + \|y\|^\alpha \Psi_0^2) \le s(\mathcal{L}_0),$$

and so necessarily $s(\mathcal{L}_0) > 0$. This can be also shown from the abstract theory in [2] and [4]. This completes the proof. $\square$

Remark 4.2. If ∂D is of class C^1, then in Theorem 4.1 we can allow $x_0 \in \partial D$. The only difference is that the limit problem is a half space problem, instead of a full space problem.

Corollary 4.3. *Suppose $a_{ij} \in W^{1,\infty}(D)$, b_i, $c \in L^\infty(D)$, $V \in L^\infty(\mathbf{R}^N)$,*

$$\{ x \in D \ : \ V(x) = V_L \} = \{x_0, ..., x_p\}, \quad p \ge 0.$$

and for each $i = 0, ..., p$,

$$V(x) = V_L + \|x - x_i\|^{\alpha_i} g_i(\omega) + o(\|x - x_i\|^{\alpha_i}) \qquad as \ \ x \to x_i, \qquad \omega = \frac{x - x_i}{\|x - x_i\|},$$

where $\alpha_i > 0$ and $g_i : S^{N-1} \to \mathbf{R}_+$ is bounded and positive. Set

$$\alpha := \max_{0 \le i \le p} \alpha_i$$

and let $\{i_1, ..., i_q\} \subset \{0, ..., p\}$ be the set of indices for which

$$\alpha_j = \alpha, \qquad j = i_1, ..., i_q.$$

Then

(4.33)
$$\lim_{h\downarrow 0} \frac{\sigma_1^D[h^2 L + V] - V_L}{h^{\frac{2\alpha}{\alpha+2}}} = \min_{1 \le j \le q} s(\mathcal{L}_j),$$

where

$$\mathcal{L}_j = -\sum_{k,\ell=1}^N a_{k\ell}(x_{i_j}) \partial_k \partial_\ell + \|y\|^\alpha g_{i_j}(\omega), \qquad 1 \le j \le q.$$

Thanks to Theorem 4.1, (4.33) is true if $q = 1$. This corollary is an extension of Theorem 4.1 to cover the case when $q \ge 2$.

Proof. Pick $j \in \{1, ..., q\}$. Let V_j denote any potential which is equal to V within a sufficiently small neighborhood of x_{i_j}, is bounded away from V_L outside that neighborhood, and $V_j \ge V$. By the monotonicity of the principal eigenvalue with respect to the potential we have

$$\sigma_1^D[h^2 L + V] \le \sigma_1^D[h^2 L + V_j].$$

Moreover, thanks to Theorem 4.1,

$$\lim_{h\downarrow 0} \frac{\sigma_1^D[h^2 L + V_j] - V_L}{h^{\frac{2\alpha}{\alpha+2}}} = s(\mathcal{L}_j).$$

Thus,

$$\limsup_{h\downarrow 0} \frac{\sigma_1^D[h^2 L + V] - V_L}{h^{\frac{2\alpha}{\alpha+2}}} \le s(\mathcal{L}_j).$$

Since this is true for any j, the following holds:

(4.34) $$\limsup_{h\downarrow 0} \frac{\sigma_1^D[h^2 L + V] - V_L}{h^{\frac{2\alpha}{\alpha+2}}} \le \min_{1\le j\le q} s(\mathcal{L}_j).$$

To complete the proof, pick $j_0 \in \{1, ..., q\}$ such that

$$s(\mathcal{L}_{j_0}) := \min_{1\le j\le q} s(\mathcal{L}_j).$$

Then, the blowing-up scaling

$$y = h^{\frac{-2}{\alpha+2}}(x - x_{i_{j_0}}), \qquad \Psi_h(y) = h^{\frac{N}{\alpha+2}}\Omega_h(x), \qquad x \in D,$$

works out to get the first part of the proof of Theorem 4.1. Moreover, thanks to (4.34), if (4.33) fails to be true, then there exist $\lambda_0 \in \mathbf{R}$ and a sequence (h_n), $n \ge 1$, such that $\lim_{n\to\infty} h_n = 0$ and

(4.35) $$\lim_{n\to\infty} \frac{\sigma_1^D[h_n^2 L + V] - V_L}{h_n^{\frac{2\alpha}{\alpha+2}}} = \lambda_0 < s(\mathcal{L}_{j_0}).$$

Using the same argument as in the proof of Theorem 4.1, (4.35) leads to a contradiction. This completes the proof. ∎

Also, the proof of Theorem 4.1 can be adapted to cover some special cases where V_L is taken along infinitely many points. Indeed, assume that

$$\{ x \in D : V(x) = V_L \} = \{x_n, \ n \ge 1\},$$

where x_n is a sequence such that

$$\lim_{n\to\infty} x_n = x_0 \in \bar{D}.$$

Suppose in addition that for each $n \ge 1$

$$V(x) = V_L + \|x - x_n\|^n g_n(\omega) + o(\|x - x_n\|^n) \qquad \text{as } x \to x_n, \qquad \omega = \frac{x - x_n}{\|x - x_n\|},$$

where $g_n : S^{N-1} \to \mathbf{R}_+$ is bounded and positive. Then, using the same argument as in the proof of Theorem 4.1, we find that for each $n \ge 1$

(4.36)
$$\limsup_{h\downarrow 0} \frac{\sigma_1^D[h^2 L + V] - V_L}{h^{\frac{2n}{n+2}}} \le s(-\sum_{i,j=1}^N a_{ij}(x_n)\partial_i\partial_j + \|y\|^n g_n(\omega)).$$

Since $\lim_{n\to\infty} \frac{2n}{n+2} = 2$, (4.36) suggests that h^2 is the exact decay order of the principal eigenvalue to V_L. But this is not the case, as the following version of Theorem 6.4 in [16] shows.

Theorem 4.4. *Suppose $a_{ij} \in W^{1,\infty}(D)$, b_i, $c \in L^\infty(D)$, $V \in L^\infty(\mathbf{R}^N)$ and*

(4.37) $$|\{ x \in D : V(x) = V_L \}| = 0.$$

Then,

(4.38) $$\lim_{h\downarrow 0} \frac{\sigma_1^D[h^2 L + V] - V_L}{h^2} = \infty.$$

The details of the proof can be adapted mutatis mutandis from the proof of Theorem 6.4 in [16], and so we will omit them. Due to Theorem 4.4 it would be of great interest to ascertain the exact decay rate in the previous example.

5. Decay rate for the case when V vanishes on a subdomain

The main result of this section is the following.

Theorem 5.1. *Suppose $a_{ij} \in W^{1,\infty}(D)$, b_i, $c \in L^\infty(D)$ and $V \in L^\infty(D)$. Set*

$$D_+ := \{x \in D \,:\, V(x) > V_L\}$$

and suppose that

$$D_0 := D \setminus \bar{D}_+$$

is an open subset of D which is stable in the sense of Babuška and Výborný [3]. Then,

$$(5.1) \qquad \lim_{h \downarrow 0} \frac{\sigma_1^D[h^2 L + V] - V_L}{h^2} = \sigma_1^{D_0}[L]\,.$$

Moreover, if $\Omega_h > 0$ stands for the principal eigenfunction associated with $\sigma_1^D[h^2 L + V]$, normalized by

$$(5.2) \qquad \int_D \Omega_h^2 = 1\,,$$

then, for any sequence $h_n \to 0$ as $n \to \infty$, there exists a subsequence of Ω_{h_n}, relabeled by n, such that

$$(5.3) \qquad \lim_{n \to \infty} \|\Omega_{h_n} - \hat{\Omega}_0\|_{L^2(D)} = 0\,,$$

where $\hat{\Omega}_0$ is the extension by zero of a principal eigenfunction Ω_0 associated with $\sigma_1^{D_0}[L]$ normalized by

$$\int_{D_0} \Omega_0^2 = 1\,.$$

Relation (5.1) was found in Theorems 6.2, 6.7 of [16] for a less general class of operators and subdomains. The behavior of the ground states is new. Our proof of (5.3) will provide us simultaneously with a striking proof of (5.1). We should point out that since D_0 is not assumed to be connected, the principal eigenvalue is not necessarily simple. This is why $\hat{\Omega}_0$ is the extension by zero of some of its associated eigenfunctions. If D_0 is connected, then $\hat{\Omega}_0$ is uniquely determined and

$$\lim_{h \downarrow 0} \|\Omega_h - \hat{\Omega}_0\|_{L^2(D)} = 0.$$

Proof of Theorem 5.1. Thanks to Theorem 3.1, we can assume

$$V_L = 0\,.$$

The function Ω_h satisfies

$$(5.4) \qquad -\sum_{i,j=1}^N a_{ij} \partial_i \partial_j \Omega_h + \sum_{j=1}^N b_j \partial_j \Omega_h + c \Omega_h^2 + \frac{V}{h^2} \Omega_h = \sigma_1^D\Big[L + \frac{V}{h^2}\Big] \Omega_h$$

in D, and $\Omega_h = 0$ on ∂D. Multiplying (5.4) by Ω_h, integrating on D, applying the formula of integration by parts and using (5.2) gives

(5.5)
$$\sum_{i,j=1}^{N} \int_D a_{ij} \partial_i \Omega_h \partial_j \Omega_h + \sum_{j=1}^{N} \int_D B_j \Omega_h \partial_j \Omega_h + \int_D c\Omega_h + \int_D \frac{V}{h^2}\Omega_h^2 = \sigma_1^D[L + \frac{V}{h^2}],$$

where the new coefficients B_j are those given by (3.10). Since D_0 is a proper subdomain of D and $V = 0$ on D_0, we have

(5.6)
$$\sigma_1^D[L + \frac{V}{h^2}] < \sigma_1^{D_0}[L].$$

In particular,

(5.7)
$$\limsup_{h\downarrow 0} \sigma_1^D[L + \frac{V}{h^2}] \le \sigma_1^{D_0}[L].$$

Moreover, by the Hölder's inequality

(5.8)
$$|\sum_{j=1}^{N} \int_D B_j \Omega_h \partial_j \Omega_h| \le \beta \|\nabla\Omega_h\|_{L^2(D)},$$

where

$$\beta := \sup_{x\in D} \|B(x)\|, \qquad B = (B_i).$$

Thus substituting (5.6) and (5.8) into (5.5) gives

$$\nu\|\nabla\Omega_h\|_{L^2(D)}^2 - \beta\|\nabla\Omega_h\|_{L^2(D)} + \int_D \frac{V}{h^2}\Omega_h^2 \le \sigma_1^{D_0}[L] - c_L,$$

where $c_L := \text{ess inf } c$ and $\nu > 0$ is the ellipticity constant of L. From this inequality it is easily seen that there exists a constant $K > 0$ such that

(5.9)
$$\|\nabla\Omega_h\|_{L^2(D)} \le K, \qquad \int_D \frac{V}{h^2}\Omega_h^2 \le K, \qquad \forall h > 0.$$

Moreover, going back to (5.5) and using (5.9), the same argument as above also shows that there exists $M \in \mathbf{R}$ such that

(5.10)
$$\sigma_1^D[L + \frac{V}{h^2}] \ge M \qquad \forall h > 0.$$

We now show (5.1). The argument will follow by contradiction. If (5.1) is not true, by (5.7) and (5.10) there exist $\lambda_0 \in \mathbf{R}$ and a sequence (h_n), $n \ge 1$, such that $\lim_{n\to\infty} h_n = 0$ and

(5.11)
$$\lim_{n\to\infty} \sigma_1^D[L + \frac{V}{h_n^2}] = \lambda_0 < \sigma_1^{D_0}[L].$$

Moreover, thanks to (5.9) the sequence (h_n) can be chosen so that

(5.12)
$$\lim_{n\to\infty} \Omega_{h_n} = \Psi_0 \in W^{1,2}(D) \qquad \text{weakly in } W^{1,2}(D),$$

and strongly in $L^2(D)$, for some Ψ_0. It is clear that $\Psi_0 \ge 0$ and that

$$\int_D \Psi_0^2 = 1.$$

By Theorem 3.2, $\Psi_0 = 0$ in D_+. Thus, since D_0 is stable,

$$\Psi_0 \in W_0^{1,2}(D_0).$$

To prove that Ψ_0 is a distributional solution, pick $\varphi \in C_0^\infty(D_0)$. Multiplying the equation of Ω_{h_n} by φ, integrating on D and applying the formula of integration by parts gives

$$\sum_{i,j=1}^N \int_{D_0} a_{ij}\partial_i\Omega_{h_n}\partial_j\varphi + \sum_{j=1}^N \int_{D_0} B_j\varphi\partial_j\Omega_{h_n} + \int_{D_0} c\varphi\Omega_{h_n} = \sigma_1^D[L + \frac{V}{h_n^2}]\int_{D_0} \varphi\Omega_{h_n}\,.$$

Thus, passing to the limit as $n \to \infty$, we find from (5.11) that

$$\sum_{i,j=1}^N \int_{D_0} a_{ij}\partial_i\Psi_0\partial_j\varphi + \sum_{j=1}^N \int_{D_0} B_j\varphi\partial_j\Psi_0 + \int_{D_0} c\varphi\Psi_0 = \lambda_0\int_{D_0} \varphi\Psi_0\,.$$

Therefore, $\Psi_0 \in W_0^{1,2}(D_0)$ is a weak solution of

$$L\Psi_0 = \lambda_0\Psi_0$$

in D_0 subject to homogeneous Dirichlet boundary conditions. Thanks to Theorem 2.1 this is impossible, since we are assuming that $\lambda_0 < \sigma_1^{D_0}[L]$. Therefore, (5.1) is true, and the same argument as before completes the proof of the remaining assertions of the theorem. $\square$

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